HISTORIA MATHEMATICA 12 (1985), 101–106

Additive Progression in Prehistoric Mathematics: A Conjecture

KARL M. PETRUSO

Boston University, Boston, Massachusetts 02215

This paper presents intriguing archaeological evidence that the practical properties of additive progression were recognized in the eastern Mediterranean during the Late Bronze Age (ca. 1200 B.C.). The evidence is in the form of a set of stone balance weights excavated in the 1960s from a small cargo ship that sank off the coast of southern Turkey. It is argued that a Fibonacci-like series of integers is represented in the masses of these prehistoric items, and it is demonstrated that the manner in which the balance weights were manufactured was simple, precise, and logical. © 1985 Academic Press, Inc.

Ce travail met en évidence un point archéologique intéressant, que les propriétés de la progression arithmétique étaient connues dès la fin de l'Age de Bronze (circa 1200 av. J.C.) dans les pays orientaux de la Méditerranée. Cette évidence nous vient sous forme d'une série de tares en pierre trouvées dans un petit cargo naufragé au large des cotes de la Turquie lors de fouilles effectuées dans les années 1960. Nous constatons que les poids de ces objets préhistoriques forment une suite de Fibonacci et nous montrons que ces tares furent fabriquées d'une manière à la fois simple, précise et logique. © 1985 Academic Press, Inc.

Dieser Aufsatz legt interessante archäologische Beweise dafür vor, daß die praktischen Eigenschaften der additiven Reihe im östlichen Mittelmeer schon um ca. 1200 vor Christus im späten Bronzezeitalter bekannt waren. Als Zeugnis dient ein Satz von Steingewichten, die in den sechziger Jahren aus einem kleinen, vor der Küste der südlichen Türkei gesunkenen Frachtschiff geborgen wurden. Es wird ausgeführt, daß es sich dabei um eine Fibonacci-artige Folge ganzzahliger Gewichte handelte; auch wird gezeigt, daß die Herstellung dieser vorgeschichtlichen Gegenstände auf einfache, präzise und logische Weise erfolgte. © 1985 Academic Press, Inc.

1. INTRODUCTION

The possibility that the principle of generating an additive series (in which $x_n = x_{n-1} + x_{n-2}$) had been discovered and exploited as early as the second millennium B.C. has been proposed by several scholars on the basis of investigations of the architectural designs of ancient buildings (e.g., [Badawy 1965; Preziosi 1970]). It is the purpose of the present paper to bring to the attention of historians of mathematics an unrelated set of archaeological artifacts from the ancient eastern Mediterranean which arguably exhibit the curious additive property of Fibonacci generation. It must be emphasized here that the conclusions presented in this paper are preliminary and tentative, and await corroboration from other quarters of the archaeological record.

Our understanding of the mathematical abilities and proclivities of ancient peoples is most complete for those periods and cultures that left readable documents, whether didactic (e.g., the Rhind and other Egyptian mathematical papyri), theoretical (Greek Pythagorean and Euclidean texts), or mundane (account tablets, tax and tribute assessments, inventories, etc. [Neugebauer 1969;

0315-0860/85 $3.00

Gillings 1972]). If we wish to appreciate the subtleties of mathematical achievement attained by preliterate or prehistoric peoples, however, we are at something of a disadvantage. Our main source of evidence is archaeological and is necessarily fragmentary. Metrical artifacts such as graduated linear rules, sets of clay pots manufactured to established and precisely regulated ratios of capacity, and balance weights provide tantalizing information on how ancient peoples quantified their products. These artifacts directly reflect the numerical systems recognized by their owners. Binary, decimal, duodecimal, vigesimal, and sexagesimal numerical substrata all are attested at various times and places in the metrical systems of the ancient Old World [Petrie 1926].

Balance weights are the most ubiquitous, best-preserved, and mathematically most informative metrical artifacts that survive to us from antiquity. A well-preserved set of stone balance weights, even if they bear no denominational markings, can reveal the absolute mass on which their system was based, in addition to the mathematical system of ratios that governed their use. For example, a hypothetical set of six balance weights from a closed (i.e., uncontaminated) archaeological deposit, whose members are deemed to be very close to their originally intended masses, weighing 7.1, 35.5, 72.3, 139.8, 348.4 and 702.6 grams, might be recognized as, respectively, 1-, 5-, 10-, 20-, 50-, and 100-unit denominations of an absolute mass in the vicinity of 7 grams (allowances must be made, of course, for imprecision due to the weighing technology available in antiquity; on this, see especially [Skinner 1954; Berriman 1955]). Such a series of weights, whether its standard absolute mass was ca. 3.5, 7, or 14 grams, would still be clearly recognizable as decimal in its structure.

This report concerns a group of haematite balance weights of a shape referred to in the archaeological literature as "sphendonoid," from the ancient Greek *sphendónē*, meaning sling bullet (i.e., approximately the shape of an olive pit; see Fig. 1). These weights were discovered in the underwater excavation of a small cargo ship that sank in about 30 meters of water around 1200 B.C. near Cape Gelidonya (modern Finike) off the south coast of Turkey. The director of the excavation, Dr. George Bass, published in the final excavation report a detailed metrological analysis of some sixty balance weights, fashioned in seven different shapes, which were found on board the vessel [Bass 1967, 135–142]. He was unable to discern any connection between the shapes of the balance weights and the systems on which they were based, although he argued that several national systems of weight, including standards known to have been in use in Egypt, Syria, Palestine,

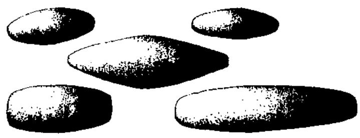

FIGURE 1

and other places in the eastern Mediterranean at this period (the Late Bronze Age) were represented on board the ship.

My recent metrological examination of the balance weights from the Cape Gelidonya merchantman has led me to conclude that the crew—which, to judge from the cargo excavated by Bass, was a group of itinerant metalsmiths whose ports of call were spread widely about the eastern Mediterranean—was able to transact business on a rather smaller range of different systems of weight mensuration than Bass had suspected. I believe that the shapes of the balance weights used by the crew were, in fact, indicative of separate systems. I have presented elsewhere [Petruso 1981, 1984] a more detailed analysis of the metrological significance of these pieces; what concerns us here is only the set of sphendonoids. All but one are well preserved and close to their original masses. Their published masses, in grams, are given in Table I.

2. METROLOGICAL ANALYSIS

The lightest piece in the group bears a single incised stroke which likely signifies its denomination (i.e., 1) as the standard unit on which the series was based. If we accept this evaluation as a working hypothesis, which is rendered likely by the fact that it is in the range of both the Egyptian *qedet* and the Syrian *necef* (either of which would have been an appropriate and useful standard unit for a merchant traveling in this region), the larger denominations fall into line quite precisely (see Table II).

It will be noted that the deviations from the proposed theoretical (i.e., predicted) masses are quite evenly distributed (six are higher, six lower than their respective theoretical masses), suggesting that our hypothetical standard of 9.3 grams is close indeed to the mass on which the system was based. Moreover, the deviations expressed in percentages fall, without exception, comfortably within the range of tolerance in precision allowed by Bronze Age weighers.

The Fibonacci series is generated from zero by adding two adjacent members

TABLE I
MASSES[a] OF SPHENDONOID BALANCE WEIGHTS FROM THE CAPE GELIDONYA SHIPWRECK

9.3	63.9	109.5
28.0	65.5	284.5
45.5	65.5[b]	468.0
46.0[b]	66.5	501.0
	67.5	

[a] In grams.
[b] Approximate mass; slightly chipped and underweight. Mass is that estimated by the excavator to have been originally intended.

TABLE II
DENOMINATIONS PROPOSED FOR GELIDONYA BALANCE WEIGHTS

Actual mass	Proposed denomination	Theoretical mass	Actual–theoretical discrepancy (%)
9.3	1	9.3	0.0
28.0	3	27.9	+0.4
45.5	5	46.5	−2.1
46.0	5	46.5	−1.1
63.9	7	65.1	−1.8
65.5	7	65.1	+0.6
65.5	7	65.1	+0.6
66.5	7	65.1	+2.2
67.5	7	65.1	+3.7
109.5	12	111.6	−1.9
284.5	31	288.3	−1.3
468.0	50	465.0	+0.6
501.0	54	502.2	−0.2

(i.e., 0; 0 + 1 = 1; 1 + 1 = 2; 1 + 2 = 3; 2 + 3 = 5; 3 + 5 = 8; . . . *ad infinitum*). The peculiar properties of Fibonacci generation and its manifestations in nature have been the subject of much scholarly interest since they were recognized in recent times by the mathematician Leonardo of Pisa (a.k.a. Filius Bonaccio) in the 13th century (on Fibonacci numbers in general, see [Brousseau 1965; Hoggatt 1972]). If we compare the Cape Gelidonya series of integers with the basic Fibonacci progression, we will notice both similarities and differences (see Table III).

No sphendonoids answering to the missing values of 2 and 19 units (predicted masses ca. 18.6 and 176.7 grams, respectively) were recovered from the seabed, but we may with confidence restore them to the crew's set of balance weights. While the smaller of these would be a useful denomination in any set of weights, the larger is at first glance awkward; in general, as Petrie observed [1926, 7] we ought not expect to find in ancient (or modern) metrical artifacts cumbersome primes above 5. In this particular case, though, the undeniable presence of a cluster at 7 units cannot be overlooked, especially since, when combined in a balance pan with the next-smaller denomination, the 7-unit weight produces the logical and useful 12-unit denomination.

TABLE III
COMPARISON OF FIBONACCI AND GELIDONYA SERIES

Fibonacci	(1)	1	2	3	5	8	13	21	34	55	89	. . .
Gelidonya		1	—	3	5	7	12	—	31	50	54	

I propose that we are dealing here with a purposeful and quite utilitarian shift in the basic Fibonacci series, from the normal 8 to 7 units in the sixth integer position. Such a shift—easily executed by the manufacturer of a set of balance weights—would allow the generation of a 50-unit (rather than 55-unit) mass farther along the series, which would be an eminently desirable denomination for the evaluation and tallying of larger quantities of commodities assayed and trafficked by weight. The principles and attractions of decimal factoring had been long since known in the Near East and Egypt, so we should not be surprised to see them manifested in the metrical artifacts on board this Bronze Age version of a tramp steamer. This hypothesis requires the restoration of a 19-unit denomination at the appropriate position in the progression. While such a denomination was not encountered among the sphendonoids recovered from the ship, its existence in the set is strongly suggested—indeed, required—by both the subsequent 31-unit piece (12 + 19) and the 50-unit piece (19 + 31). It need hardly be added that 19 + 31 + 50 = 100, giving the building block for the next order of magnitude of weighing, which approximates our modern kilogram. I must admit that the presence of what appears to be a 54-unit weight is unexpected, and I have no ready explanation for it. It has no apparent function in the additive series we have posited, although it might well have had a specific, idiosyncratic (industrial) purpose which is now lost to us.

3. CONCLUSIONS

This is not the place to consider the epistemological problem of gaps in the archaeological record due to the fragmentary nature of the data; that is a large and special issue with its own massive bibliography. It is admittedly impossible to prove the former existence of the missing denominations in this proposed progression, but the overall metrical and mathematical pattern betrayed by the pieces that do survive to us is quite clear, and, in its way, sensible as well. Our modern approach to double-pan weighing is conceptually slightly different. A set of brass metric weights, for instance, might today comprise pieces in the denominations 1, 2, 5, 10, 25, 50, 100, 250, 500, and 1000 grams. Persons who frequently work with a beam balance in shops and laboratories learn intuitively how to use weights efficiently and quickly, and become quite adept at adding and counterbalancing weights to produce a horizontal beam. The mass of an object in the left pan will equal the sum of the masses of brass weights in the right, minus the sum of the weights (if any) added to the object in the left pan. Surely the procedure was identical in the Bronze Age; the mathematical structure of the system, however, could be designed to suit the preferences and mental dexterity of the weigher. One possible advantage to having a Fibonacci-generated set of denominations is that one could get along comfortably with a small and manageable total number of pieces whose accuracy could be checked whenever necessary, using nothing more sophisticated than the double-pan balance itself. While denominations of 19 and 31 might seem to us inconvenient, one could quickly become facile at using them.

It seems that the additive series of weights hypothesized for the Cape Gelidonya ship was somewhat peculiar vis-à-vis other excavated sets of weights of the period. To judge from the total corpus of comparable material from the ancient Old World, our assessment herein of the Gelidonya sphendonoids as metrical oddities is warranted; by far the majority of systems recovered elsewhere from the ancient world betray denominations at simple and (to us, at least) more logical and elegant denominations. On board the Gelidonya vessel, for example, Bass [1967, 139] argued for the existence of the Egyptian *qedet* in a number of different shapes and denominations. I have been able to demonstrate that the dome-shaped haematite weights were scaled on a simple decimal system [Petruso 1984]. Aside from the sphendonoid and domed pieces, no other shape survives in numbers large enough to permit the metrological attribution of its representatives with any statistical confidence.

The early history of weight metrology in the eastern Mediterranean is far from understood in its entirety, and it is to be hoped that future excavation of similar mundane but subtle metrical artifacts might enable us more fully to comprehend a profoundly important, if thus far perplexing, stage in the refinement of man's perception of number.

REFERENCES

Badawy, A. 1965. *Ancient Egyptian architectural design: A study of the harmonic system.* Berkeley/Los Angeles: Univ. of California Press.

Bass, G. 1967. Cape Gelidonya: A Bronze Age shipwreck. *Transactions of the American Philosophical Society* **57**(8).

Berriman, A. 1955. Some marked weights in the Petrie collection. *Journal of Egyptian Archaeology* **41,** 48–50.

Brousseau, A. 1965. *Introduction to Fibonacci's discovery.* Santa Clara, Calif.: Fibonacci Association.

Gillings, R. 1972. *Mathematics in the time of the pharaohs.* Cambridge, Mass.: MIT Press.

Hoggatt, V. 1972. *Fibonacci and Lucas numbers.* Santa Clara, Calif.: Fibonacci Association.

Neugebauer, O. 1969. *The exact sciences in antiquity.* New York: Dover.

Petrie, Sir W. M. Flinders. 1926. *Ancient weights and measures.* London: British School of Archaeology in Egypt.

Petruso, K. 1981. Balance weights from the Cape Gelidonya shipwreck: A re-examination. Abstract of a paper read at the annual meeting of the Archaeological Institute of America, New Orleans. *American Journal of Archaeology* **85,** 211.

Petruso, K. 1984. Prolegomena to late Cypriot weight metrology. *American Journal of Archaeology* **88,** 293–304.

Preziosi, D. 1970. *Labrys: Report on architectural and settlement analysis at Yale University.* New Haven, Conn.: Yale Univ. Press.

Skinner, F. 1954. Measures and weights. In *A history of technology,* C. Singer et al., eds., Vol. I, pp. 774–784. London: Oxford Univ. Press.